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Non-monotonous behavior of the number variance, Mandel factor, invariant uncertainty product and purity for the quantum damped harmonic oscillator

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A B S T R A C T

We consider the relaxation of the quantum harmonic oscillator in the framework of the standard master equation. Analyzing the known solutions, we have found some new interesting effects. Namely, we show that the evolution of some parameters, such as the energy fluctuations, the Mandel factor, the invariant uncertainty product and quantum purity, can exhibit non-monotonous behavior with big deviations from the initial and final values. Also, we show that highly excited initial Fock states maintain their non-classical nature longer than low excited ones. During the evolution, the quantum purity of these states stays at almost constant low level during a long period, after the fast decrease initially and before the fast return to the final value at the end.

1. Introduction

We consider the standard master equation for the quantum damped harmonic oscillator in the form

$$\frac{d\hat\rho}{dt} + i\omega\left[\hat a^\dagger \hat a,\ \hat\rho\right] = \gamma(1+\nu)\left(2\hat a\hat\rho\hat a^\dagger - \hat a^\dagger\hat a\hat\rho - \hat\rho\hat a^\dagger\hat a\right) + \gamma\nu\left(2\hat a^\dagger\hat\rho\hat a - \hat a\hat a^\dagger\hat\rho - \hat\rho\hat a\hat a^\dagger\right). \qquad (1)$$

Here $\hat a$ and $\hat a^\dagger$ are the boson annihilation and creation operators, satisfying the commutation relation $\left[\hat a, \hat a^\dagger\right] = 1$, and $\hat\rho$ is the statistical operator. The parameter ν is the mean number of quanta in the thermal reservoir, ω is the oscillator eigenfrequency and γ is damping coefficient. The right-hand side of Eq. (1) is frequently called nowadays as the Lindblad operator, although this form was derived in the framework of different approaches in the mid-1960s [1–7]. Exact solutions to Eq. (1) were obtained by many authors in different representations, discrete and continuous. However, some interesting consequences of these solutions were not noticed until now. Therefore, our aim is to study that consequences, namely, the non-monotonous evolution of the quantum number variance, the Mandel factor, the invariant uncertainty product and the purity.

2. Solution to the master equation in the Fock basis: the generating function and its applications

The immediate consequence of Eq. (1) is the infinite set of linear ordinary differential equations for the diagonal elements $p_n \equiv \langle n|\hat\rho|n\rangle$ in the Fock basis, i.e., the probabilities of finding n quanta in the state $\hat\rho$,

$$\dot p_n = 2\gamma(1+\nu)\left[(n+1)p_{n+1} - np_n\right] + 2\gamma\nu\left[np_{n-1} - (n+1)p_n\right]. \qquad (2)$$

Equations (2) were derived and solved even before the operator equation (1), e.g., in Refs. [8,9].

The simplest way of solving the system (2) is to use the generating function of an auxiliary variable z,

$$G(z;t) \equiv \sum_{n=0}^{\infty} p_n(t) z^n. \qquad (3)$$

Indeed, one can easily see that this function satisfies a simple linear *first-order* partial differential equation

$$\frac{\partial G}{\partial t} = 2\gamma(1-z)[1+\nu(1-z)]\frac{\partial G}{\partial z} - 2\gamma\nu(1-z)G, \qquad (4)$$

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so that it can be easily solved in the general form [7,9–12] (particular cases of the initial thermal, coherent, and Fock states were also considered in [13])

$$G(z,u) = [1+\nu u(1-z)]^{-1} G_0\left(\frac{z+u(1+\nu)(1-z)}{1+\nu u(1-z)}\right), \tag{5}$$

where $G_0(z) \equiv G(z,0)$ and $u \equiv 1-\exp(-2\gamma t)$. Putting $z=1$ in (5) we verify the normalization condition $G(1,t) \equiv 1$. Neither the frequency ω nor the Planck constant enter formula (5).

The derivatives of the generating function with respect to parameter z, taken at $z=1$, give the statistical moments of the number operator $\hat{n} \equiv \hat{a}^\dagger \hat{a}$. In particular,

$$\langle \hat{n} \rangle = \partial G/\partial z|_{z=1}, \quad \langle \hat{n}(\hat{n}-1)\rangle = \partial^2 G/\partial z^2\Big|_{z=1}. \tag{6}$$

Using (5) and (6), one obtains the time-dependent mean number of quanta

$$\langle \hat{n} \rangle(u) = \nu u + n_0(1-u), \quad n_0 \equiv \langle \hat{n} \rangle(0). \tag{7}$$

This well known formula shows a monotonous transition from the initial value n_0 to the final value ν. But the behavior of the higher order moments can be more interesting, and this is the subject of our study. The simplest examples are the photon number variance

$$\sigma = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 = \left[\partial^2 G/\partial z^2 - (\partial G/\partial z)^2 + \partial G/\partial z\right]_{z=1}, \tag{8}$$

and the Mandel factor [14]

$$Q \equiv \frac{\langle \hat{n}(\hat{n}-1)\rangle - \langle \hat{n} \rangle^2}{\langle \hat{n} \rangle} = \frac{\partial^2 G/\partial z^2 - (\partial G/\partial z)^2}{\partial G/\partial z}\Bigg|_{z=1}, \tag{9}$$

which is frequently used to distinguish between the sub-Poissonian ($Q<0$), Poissonian ($Q=0$) and super-Poissonian ($Q>0$) statistics. Using (5) and (6), we find (with $\sigma(0) \equiv \sigma_0$)

$$\sigma(u) = \nu u(\nu u+1) + (1-u)^2\sigma_0 + n_0 u(1-u)(1+2\nu), \tag{10}$$

$$Q(u) = \frac{(\nu u)^2 + n_0(1-u)\left[(1-u)Q(0) + 2\nu u\right]}{\nu u + n_0(1-u)}. \tag{11}$$

The evolution of these quantities depends not only on their own initial and final values, but on the initial mean value n_0, too.

3. Evolution of the photon number variance

The behavior of function $\sigma(u)$ in the interval $0<u<1$ can be understood, if one looks at its derivatives at $u=0$ and $u=1$:

$$(d\sigma/du)_{u=0} = \nu - 2\sigma_0 + n_0(1+2\nu), \tag{12}$$

$$(d\sigma/du)_{u=1} = (\nu - n_0)(1+2\nu). \tag{13}$$

Comparing (13) with the difference between the initial and final values, $\sigma_0 - \nu(\nu+1)$, we conclude that $\sigma(u)$ is non-monotonous function, crossing the level $\nu(\nu+1)$, under the condition

$$[\sigma_0 - \nu(\nu+1)](n_0 - \nu) < 0. \tag{14}$$

The extremal values

$$\sigma_{ext} = \frac{4\sigma_0\nu(\nu+1) - [\nu + n_0(1+2\nu)]^2}{4\left[\nu^2 + \sigma_0 - n_0(1+2\nu)\right]} \tag{15}$$

are achieved at

$$u_e = \frac{2\sigma_0 - \nu - n_0(1+2\nu)}{2\left[\sigma_0 + \nu^2 - n_0(1+2\nu)\right]}. \tag{16}$$

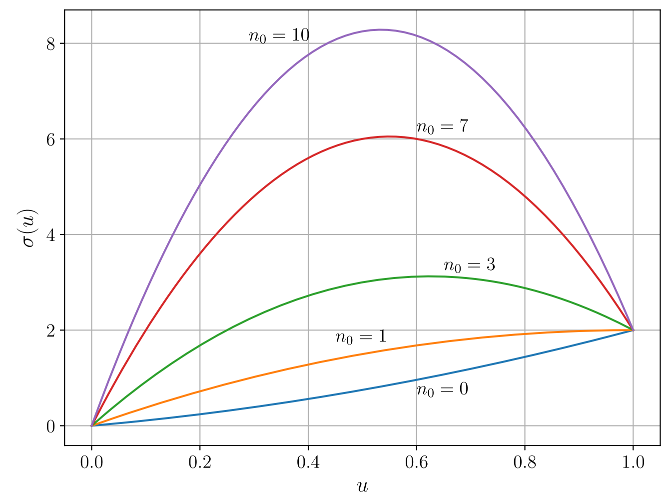

Fig. 1. Function $\sigma(u)$ for the initial Fock states with different values of n_0 and $\nu = 1$.

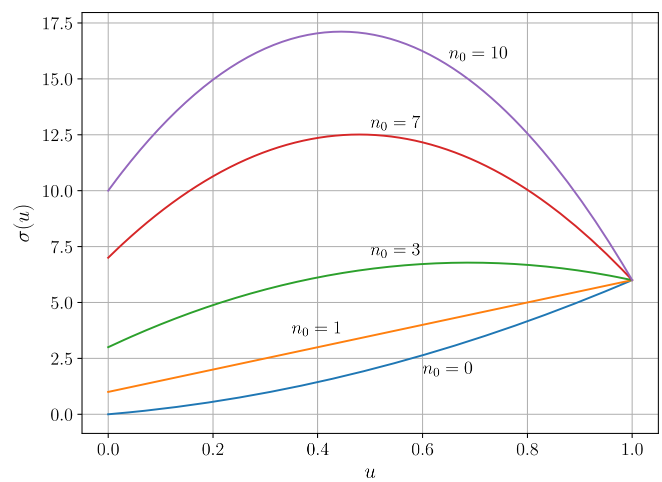

Fig. 2. Function $\sigma(u)$ for the initial coherent states with different values of n_0 and $\nu = 2$.

In particular, the variances of highly excited initial Fock states, with $\sigma_0 = 0$ and $n_0 \gg \nu$, can achieve very big values $\sigma_{max}^{(Fock)} \approx n_0(1+2\nu)/4$ at $u_e \approx 1/2$, before going to the final value $\nu(\nu+1)$. Note that at this instant we have $\langle \hat{n} \rangle(u_e) \approx n_0/2$. A typical example is shown in Fig. 1.

An interesting case is the initial coherent state with $\sigma_0 = n_0$. Then condition (14) means that $\nu < n_0 < \nu(\nu+1)$. But the maximum continues to exist for $n_0 > \nu(\nu+1)$, provided $\nu > 1/2$. In this case, $\sigma(u) \ge \nu(\nu+1)$ in the whole interval. The maximum can be as high as for the initial Fock states, if $n_0 \to \infty$, but its position is shifted to $u_m \approx (2\nu-1)/(4\nu)$. A typical example is shown in Fig. 2.

If $n_0 = \nu$, then we see the monotonous transition

$$\sigma(u) = \nu(\nu+1) + [\sigma_0 - \nu(\nu+1)](1-u)^2.$$

On the other hand, if $\sigma_0 = \nu(\nu+1)$, then

$$\sigma(u) = \sigma_0 + (1+2\nu)(n_0-\nu)u(1-u),$$

so that the extremum is achieved exactly at $u_e = 1/2$, with $\sigma_{ext} = [\nu(3+2\nu) + n_0(1+2\nu)]/4$ being maximum for $n_0 > \nu$ and minimum for $n_0 < \nu$.

If $n_0 \ll \nu$ but $\sigma_0 \gg \nu(\nu+1)$, then function $\sigma(u)$ has the minimum when u is close to unity, but this minimum is only slightly below the final value $\nu(\nu+1)$. Such a situation can happen, e.g., for the simple initial superposition

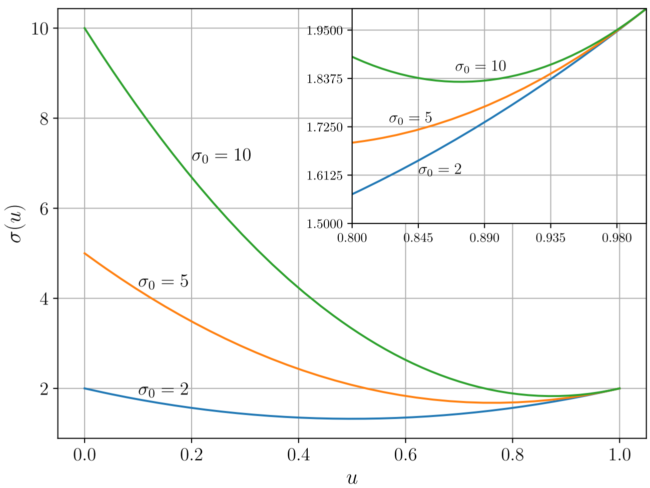

Fig. 3. Function $\sigma(u)$ for $\nu = 1$, $n_0 = 0.1$ and different values of σ_0.

$$|\psi\rangle = \sqrt{1-\varepsilon}|0\rangle + \sqrt{\varepsilon}|N\rangle, \quad \varepsilon \ll 1, \quad N \gg 1, \tag{17}$$

when $n_0 = \varepsilon N$ but $\sigma_0 = \varepsilon(1-\varepsilon)N^2$. Although the initial state is "almost vacuum" in this example, the behavior of the variance is totally different from the case of the initial exact vacuum state. Fig. 3 gives some illustrations.

4. Evolution of the Mandel factor

The evolution of Q-factor from the initial value Q_0 to the final thermal value $Q_f = \nu$ is very simple in the case of zero-temperature reservoir with $\nu = 0$, when $Q(u) = Q_0(1-u)$, so that $Q(u)$ goes to zero without changing its sign. But if $\nu > 0$, then the evolution can be more interesting.

It can be convenient to rewrite Eq. (11) as

$$\tilde{Q}(u) = \frac{u^2 + a(1-u)\left[(1-u)q + 2u\right]}{u + a(1-u)}, \quad \tilde{Q} \equiv Q/\nu, \tag{18}$$

where $a = n_0/\nu$ and $q = Q_0/\nu > 0$. For the initial thermal state with $Q_0 = n_0$ we have $q = a$ and the monotonous behavior $Q(u) = \langle\hat{n}\rangle(u)$, in accordance with the property of the thermal state to remain thermal with time dependent temperature. Also, the evolution is monotonous, if $a = 1$: $\tilde{Q}(u) = (q-1)(u-1)^2 + 1$.

Since $\left(d\tilde{Q}/du\right)_{u=1} = 1 - a$, for any value of q, function $Q(u)$ possesses extremal values (maximum for $q < 1$ or minimum for $q > 1$) under the condition

$$(a-1)(q-1) < 0. \tag{19}$$

Looking for extremal values, we arrive at a quadratic equation with respect to the variable u, which gives two solutions,

$$u_e = \frac{-aB \pm \sqrt{aB(q-a)}}{B(1-a)}, \tag{20}$$

where $B = 1 + aq - 2a = 1 - a + a(q-1)$. If $q > 1$, then the extremum (minimum) exists for $a < 1$, due to the condition (19). In such a case, $B > 0$, as well as $q - a = q - 1 + 1 - a$. Consequently, the argument of the square root in (20) is positive. Moreover, we have to choose the positive sign in front of the square root, in order to have positive value of u_e. Similar reasonings show that for $q < 1$ we have $B < 0$ and $q - a < 0$. Then the argument of the square root is positive again. In this case, $a > 1$, so that $B(1-a) > 0$, but we have to choose *negative* sign in front of the square root, in order to obtain the solution satisfying the condition $u_e < 1$. Thus, we obtain the following extremal values of the ratio $\tilde{Q} = Q/\nu$:

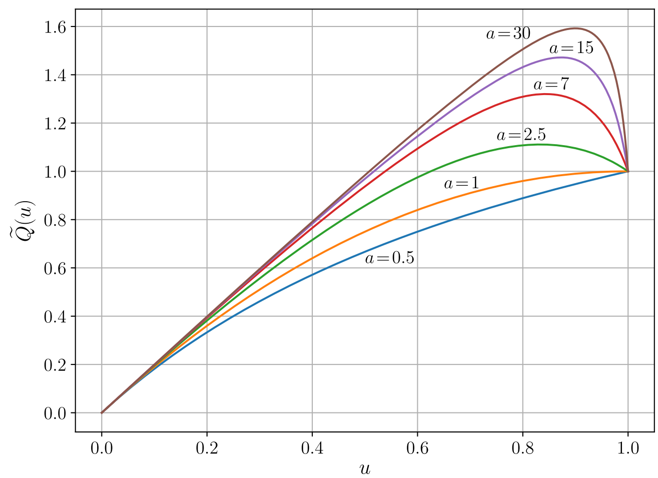

Fig. 4. Function $\tilde{Q}(u)$ for the initial coherent states with different values of parameter a.

$$\tilde{Q}_{max}\Big|_{q<1} = \frac{2\left[a(a-q) - \sqrt{a(1+aq-2a)(q-a)}\right]}{(1-a)^2}, \tag{21}$$

$$\tilde{Q}_{min}\Big|_{q>1} = \frac{2\left[a(a-q) + \sqrt{a(1+aq-2a)(q-a)}\right]}{(1-a)^2}. \tag{22}$$

If $q = 1$, then we find

$$\tilde{Q}_{ext}\Big|_{q=1} = \frac{2\sqrt{a}}{1+\sqrt{a}}, \quad u_e = \frac{\sqrt{a}}{1+\sqrt{a}}. \tag{23}$$

In this case, the normalized Mandel factor can rapidly attain a deep minimum if $a \ll 1$, but its maximum cannot be higher than 2 if $a \gg 1$. When $Q(u)$ attains the extremal value, the mean number of quanta equals $\langle\hat{n}\rangle(u_e) = \sqrt{n_0/\nu}$.

Note that condition (19) corresponds to the situation when function $\tilde{Q}$ intersects the line $\tilde{Q} = 1$ at $u < u_e < 1$ and attains a maximal or minimal value later at $u = u_e$. However, the example (23) shows that the maximal or minimal values can exist even without the intersection with the line $\tilde{Q} = 1$. This is the consequence of the expression $\left(d\tilde{Q}/du\right)_{u=0} = 2 - q(1+a)/a$. It shows that the sub-Poissonian states (with $q < 0$) cannot become "more sub-Poissonian" during the evolution. But initial super-Poissonian states with $1 < q < 2a/(1+a)$ can become "more super-Poissonian" for some interval of time.

4.1. Initial coherent states

For the initial Poissonian statistics (the simplest example is the coherent state), the behavior of the ratio Q/ν is determined by the ratio $a = n_0/\nu$ only:

$$\tilde{Q}(u) = \frac{u\left[u + 2a(1-u)\right]}{u + a(1-u)}. \tag{24}$$

Fig. 4 shows function $\tilde{Q}(u)$ for different values of parameter a. We see that this function is monotonous for $a < 1$, but it has a maximum for $a > 1$:

$$\tilde{Q}_{max} = \frac{2a\left(a - \sqrt{2a-1}\right)}{(1-a)^2}, \quad a > 1. \tag{25}$$

This maximum is achieved at

$$u_m = \frac{a}{a-1}\left(1 - \frac{1}{\sqrt{2a-1}}\right). \tag{26}$$

Note that $\tilde{Q}_{max}$ grows monotonously as function of a, from the unit value at $a = 1$ to the maximal value 2 for $a \to \infty$. On the other

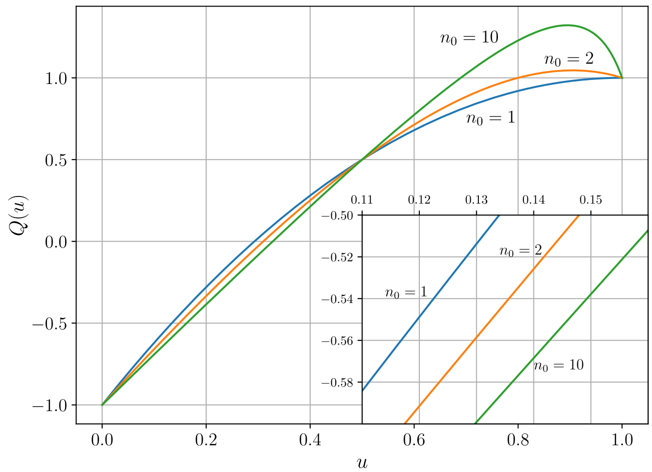

Fig. 5. The dependence $Q(u)$ with $\nu = 1$ for the initial Fock states. Note the inversion of the order of lines at $u = 1/2$.

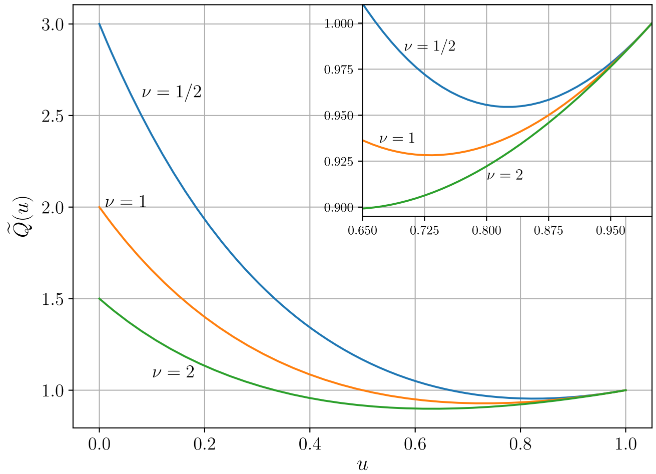

Fig. 6. The dependence $\tilde{Q}(u)$ for the initial squeezed vacuum states with $a = 1/2$ and different values of the final mean quantum number ν.

hand, the position of maximum, u_m, goes to unity for $a \to 1$ and $a \to \infty$, so that it has some minimal value. This minimal value can be found analytically: $u_m^{(min)} = 2(\sqrt{2} - 1) \approx 0.83$ for $a = 2 + \sqrt{2} \approx 3.41$. For this value of u_m we have $\tilde{Q}_{max} = 2(2 - \sqrt{2}) \approx 1.17$.

4.2. Initial sub-Poissonian states

The consequence of Eq. (18) is the partial derivative

$$\frac{\partial \tilde{Q}}{\partial a} = \frac{u(1-u)[u + q(1-u)]}{[u + a(1-u)]^2}. \tag{27}$$

This derivative is always non-negative for $q \geq 0$. In this case, the lines with different values of parameter a do not intersect inside the interval $0 < u < 1$, and $\tilde{Q}$ increases with increase of a for any fixed value of u, as we clearly see in Fig. 4.

The situation is more interesting for the so-called “non-classical” initial states with $q < 0$. Then $\partial \tilde{Q}/\partial a < 0$ at the initial stage of evolution with small values of u, while $\partial \tilde{Q}/\partial a > 0$ at the final stage. The inversion point is $u_i = |q|/(1 + |q|)$. Such a behavior of $Q(u)$ is illustrated in Fig. 5 for the “most nonclassical” initial Fock states with $Q_0 = -1$.

For initial “non-classical” states with $Q_0 < 0$, it can be interesting to know the “classicalization” (scaled) time, i.e., the value u_c when $Q(u_c) = 0$. Since the value of the normalized Mandel factor at the inversion point is positive: $\tilde{Q}_i = |q|/(1 + |q|)$ (it is curious that $\tilde{Q}_i = u_i$), we have $u_c < u_i$. This means that initial states with bigger values of n_0 maintain their “non-classical” nature longer than states with smaller values of n_0. In particular, more excited are the initial Fock states, more “robust” against the “classicalization” they are – a quite “anti-intuitive” result. The solution to the equation $Q(u_c) = 0$, satisfying the condition $0 < u_c < 1$, can be written as

$$u_c = \frac{|Q_0|}{|Q_0| + \nu\left(\sqrt{1 + |Q_0|/n_0} + 1\right)}. \tag{28}$$

It clearly shows that $\partial u_c/\partial n_0 > 0$ for all values of n_0 and $|Q_0|$. But the difference is not big. For example, the relative difference between the values of u_c for the Fock states with $n_0 = 1$ and $n_0 \to \infty$ equals $(u_{c\infty} - u_{c1})/u_{c1} = \nu\left(\sqrt{2} - 1\right)/(1 + 2\nu)$, so that it does not exceed 20% in the high temperature case $\nu \gg 1$, when $u_{c\infty} \approx (2\nu)^{-1} \ll 1$.

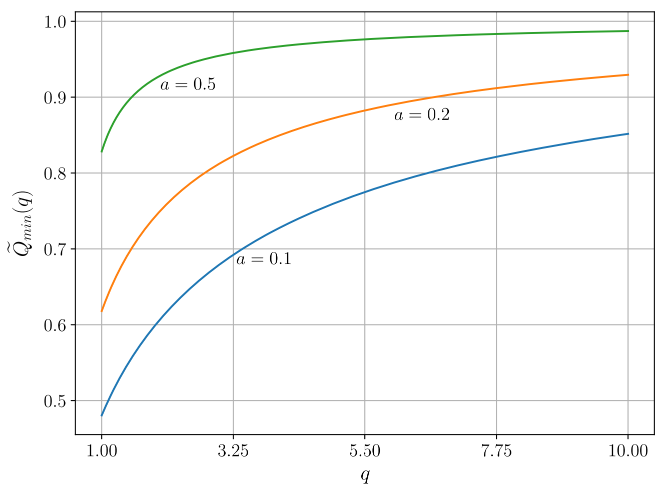

Fig. 7. The dependence $\tilde{Q}_{min}(q)$ for different values of parameter a.

4.3. Initial super- and hyper-Poissonian states

An important example of super-Poissonian states is the initial squeezed vacuum state with $Q_0 = 1 + 2n_0$. Then $q = 2a + \nu^{-1}$, and condition (19) can be satisfied for $1 > a > (\nu - 1)/(2\nu)$, with $q > 1$. Fig. 6 shows function $\tilde{Q}(u)$ for $a = 1/2$ and $\nu = 1/2, 1, 2$ (i.e., $q = 3, 2, 1.5$). Another example corresponds to so called “hyper-Poissonian” states with $Q_0 \gg 1 + 2n_0$ [15,16]. Although the minimal value (22) turns out not very far from unity for $a \ll 1$ and $aq \gg 1$, $\tilde{Q}_{min} \approx 1 - (4aq)^{-1}$, it can be appreciable for moderate values of the product aq, as shown in Fig. 7.

5. Evolution of the invariant uncertainty product

We define the dimensionless “coordinate” and “momentum” quadrature operators as

$$\hat{x} = \left(\hat{a} + \hat{a}^\dagger\right)/\sqrt{2}, \qquad \hat{p} = i\left(\hat{a}^\dagger - \hat{a}\right)/\sqrt{2}.$$

Their co-variances $\sigma_{\alpha\beta} \equiv \langle \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} \rangle/2 - \langle \hat{\alpha} \rangle \langle \hat{\beta} \rangle$ (with $\alpha, \beta = x, p$) obey the equations, which are consequences of (1),

$$\dot{\sigma}_{xx} = 2\left(\omega\sigma_{xp} - \gamma\sigma_x + \gamma\sigma_*\right), \quad \sigma_* \equiv \nu + 1/2, \tag{29}$$

$$\dot{\sigma}_{pp} = 2\left(-\omega\sigma_{xp} - \gamma\sigma_p + \gamma\sigma_*\right), \tag{30}$$

$$\dot{\sigma}_{xp} = \omega\left(\sigma_{pp} - \sigma_{xx}\right) - 2\gamma\sigma_{xp}. \tag{31}$$

The immediate consequence of Eqs. (29) and (30) is the equation for the dimensionless energy of quantum fluctuations,

$$\dot{\mathcal{E}} = 2\gamma\,(\sigma_* - \mathcal{E})\,, \qquad \mathcal{E} \equiv \left(\sigma_{xx} + \sigma_{pp}\right)/2, \tag{32}$$

whose solution is another form of Eq. (7):

$$\mathcal{E}(t) = \sigma_* u + (1-u)\mathcal{E}(0). \tag{33}$$

Another interesting consequence of all three equations, (29)–(31), is the following equation for the *Invariant Uncertainty Product* (IUP) $\Delta \equiv \sigma_{xx}\sigma_{pp} - \sigma_{xp}^2$:

$$\dot{\Delta} = 4\gamma\,[\sigma_*\mathcal{E}(t) - \Delta]\,. \tag{34}$$

Note that the oscillator frequency does not enter Eq. (34). This is the manifestation of the remarkable property of Δ: in the absence of damping, it is conserved in time for *any time-dependent quadratic Hamiltonian* [17]. Another fundamental property is the famous Schrödinger–Robertson uncertainty relation $\Delta \ge 1/4$, which holds for all quantum systems.

Using (33), we obtain the solution to Eq. (34),

$$\Delta(t) = (1-u)^2\Delta(0) + 2\sigma_*\mathcal{E}(0)u(1-u) + u^2\sigma_*^2. \tag{35}$$

It is worth emphasizing that this solution holds for *arbitrary* initial quantum states. Note also, that the initial values $\Delta(0)$ and $\mathcal{E}(0)$ are independent in the general case (satisfying the restrictions $\Delta(0) \ge 1/4$ and $\mathcal{E}(0) \ge 1/2$). In the context of our study, we wish to know, when the evolution given by Eq. (35) is monotonous or not. Simple indicators of non-monotonicity are opposite signs of the derivative $d\Delta/du$ at $u=0$ and $u=1$. Straightforward calculations result in the inequality

$$[\sigma_*\mathcal{E}(0) - \Delta(0)]\,[\sigma_* - \mathcal{E}(0)] < 0. \tag{36}$$

It is not satisfied for all initial states with the property $\sigma_{xx} = \sigma_{pp}$ and $\sigma_{xp} = 0$, such as the thermal, coherent and Fock states, because $\Delta = \mathcal{E}^2$ for such states, so that the left-hand side of (36) equals $\mathcal{E}\,(\sigma_* - \mathcal{E})^2 > 0$. Consequently, the evolution of $\Delta(t)$ is monotonous for all such states:

$$\Delta(t) = [u\sigma_* + (1-u)\mathcal{E}(0)]^2\,. \tag{37}$$

However, the evolution of Δ can be non-monotonous for pure squeezed states with $\sigma_{xx} \ne \sigma_{pp}$ and $\Delta(0) = 1/4$, provided the initial energy of fluctuations is high enough, $\mathcal{E}(0) > \sigma_*$. Some examples are shown in Fig. 8. For highly squeezed initial states with $\mathcal{E}(0) \gg \sigma_*$, the maximum of $\Delta(u)$ is achieved near the point $u = 1/2$, with $\Delta_{max} \approx \sigma_*\mathcal{E}(0)/2$.

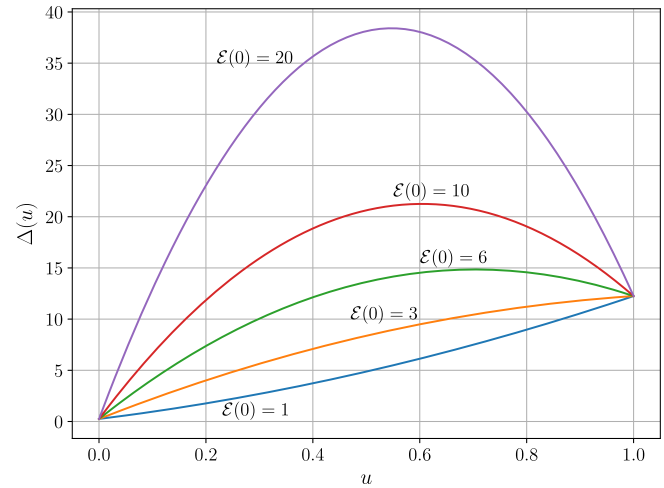

Fig. 8. The dependence $\Delta(u)$ of the invariant uncertainty product for initial pure squeezed states with different initial energies of fluctuations $\mathcal{E}(0)$; the reservoir parameter $\nu = 3$.

6. Evolution of the quantum purity

The value of the invariant uncertainty product Δ completely determines the value of quantum purity $\mu = \mathrm{Tr}\left(\hat{\rho}^2\right)$ for arbitrary *Gaussian* states, according to the formula [18,19]

$$\mu = (4\Delta)^{-1/2}. \tag{38}$$

Therefore, the purity goes monotonously from the initial value to the final one for the thermal and coherent states, but it can show a non-monotonous behavior for squeezed states. However, the behavior of $\Delta(t)$ cannot be translated to the behavior of purity for non-Gaussian states. The simplest example is the evolution of the purity of the initial Fock state $|m\rangle$ in the zero-temperature thermal reservoir ($\nu = 0$). Both the initial and final states are pure in this case, but the quantum purity can drop to very low intermediate values in the process of evolution.

The consequence of (3) and (5) with $G_0(z) = z^m$ and $\nu = 0$ is the diagonal photon distribution

$$p_n(t) = \frac{m!}{n!(m-n)!}(1-u)^n u^{m-n}. \tag{39}$$

Since the off-diagonal elements of the density matrix in the Fock basis are zero for all times in the case under study, the purity has the form

$$\begin{aligned}\mu^{Fock} &= \sum_{n=0}^{m} p_n^2(t) = u^{2m} F\left(-m, -m; 1; (1-u)^2/u^2\right) \\ &= (2u-1)^m P_m\left(\frac{1-2u+2u^2}{2u-1}\right),\end{aligned} \tag{40}$$

where $F(a,b;c;z)$ is the Gauss hypergeometric function and $P_m(z)$ the Legendre polynomial. Here we used formula 7.3.1.171 from [20],

$$F(-m,-m;1;z) = (1-z)^m P_m\left(\frac{1+z}{1-z}\right).$$

Function (40) attains its minimum at $u = 1/2$, when it equals

$$\mu_{min}^{Fock} = \frac{(2m)!}{(2^m m!)^2} = \frac{(2m-1)!!}{(2m)!!}. \tag{41}$$

For $m \gg 1$ we have $\mu_{min}^{Fock} \approx (m\pi)^{-1/2} \ll 1$. This low value is explained by the form of the photon distribution function (39) at $u = 1/2$. It is spread over the wide interval of quantum numbers $m/2 - k < n < m/2 + k$ with $1 \ll k \ll m$, in accordance with the huge value of the photon number variance $\sigma_{max}^{(Fock)} = m/4$, shown in Fig. 1. Inside this interval, $p_n^2 \approx 2/(\pi m) = const \ll 1$.

The value μ_{min}^{Fock} can be compared with the minimal value of purity $\mu_{min}^{sqz} = (1+m)^{-1/2}$ for the initial pure squeezed states with $\mathcal{E}(0) = m + 1/2$ (and $\nu = 0$). Fig. 9 shows the dependence $\mu(u)$ for the initial pure vacuum squeezed and Fock states with the same values of the initial energy $\mathcal{E}(0) = m + 1/2$ and $\nu = 0$. In this case,

$$\mu^{sqz}(u) = [1 + 4mu(1-u)]^{-1/2}\,. \tag{42}$$

Note that both functions, (40) and (42), possess the symmetry $\mu(u) = \mu(1-u)$. They demonstrate long almost flat “bottoms” in the limit $m \gg 1$. Their existence is explained by two factors: 1) the asymptotic expansion $\mu(1/2 + \varepsilon) \approx \mu_{min}\left(1 + 2\varepsilon^2\right)$ for $\varepsilon \ll 1$ and

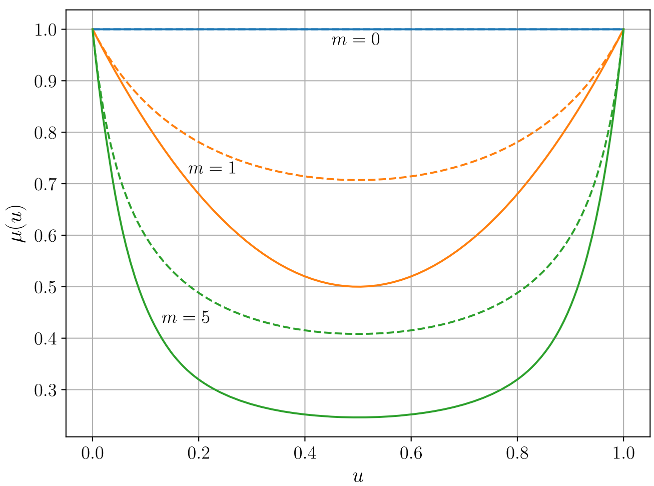

Fig. 9. The dependence $\mu(u)$ for the initial pure vacuum squeezed (dashed lines) and Fock (solid lines) states with the same values of the initial energy $\mathcal{E}(0) = m + 1/2$, for $\nu = 0$.

$m \gg 1$; 2) the equal values of the initial derivative $d\mu/du|_{u=0} = -2m$.

The non-monotonous evolution of the purity of the initial single- and two-mode squeezed states in the squeezed reservoir was discovered earlier in [19,21]. A similar behavior for the initial cat-like states (superposition of two coherent states of the form $|\alpha\rangle + \exp(i\theta)|-\alpha\rangle$ [22]) was found in [19,23,24]. However, there is an essential difference between these cases. Namely, the minimal intermediate purity of the initial Fock or squeezed states goes to zero if $\mathcal{E}(0) \to \infty$. On the other hand, although the purity of the initial cat-like states also rapidly drops to an almost constant intermediate level, this level remains above the value 1/2 even in the limit $|\alpha| \to \infty$, if $\nu = 0$ [23]. One more example of non-monotonous evolution of the purity of the initial vacuum quantum state of a selected field mode, interacting with the "bath" of all other modes in a 1D ideal cavity with oscillating walls, was given in [25]. In all that cases, long periods of almost constant intermediate values of the purity were observed.

7. Conclusions

We have demonstrated that the behavior of various quantities describing quantum fluctuations (such as the quantum number and quadrature variances, the Mandel factor, the quantum purity) during the process of standard thermal relaxation can be quite nontrivial, depending on the initial conditions. In particular, intermediate values of these quantities can be well different from the initial and final ones, attaining sometimes high maximums and sometimes deep minimums. Moreover, the example of "hyper-Poissonian" state (17) shows that, sometimes, even small changes of the quantum state can result in drastic changes of their statistical properties. Another interesting and "anti-intuitive" result is the increase of "robustness" of sub-Poissonian quantum states against the "classicalization" with the increase of the initial mean number of quanta. In addition, we have demonstrated that different measures of "non-classicality" do not correlate in the general case. For example, the initial Fock states in the zero temperature reservoir are strongly mixed during a long interval of time, but they preserve the negative value of the Mandel Q-factor. On the other hand, the initial squeezed states also show a low value of quantum purity during the same long time intervals, but they can remain highly super-Poissonian during these intervals. Note that the evolution of three quantities, the photon number variance, the Mandel Q-factor, and the invariant uncertainty product, is completely determined by their own initial values and the initial value of the mean photon number (or the mean energy). On the contrary, the evolution of the quantum purity depends on many factors, characterizing the initial quantum state.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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